SERIES WORKSHEET 3 SOLUTION SKETCHES

Note: These are not model solutions, but only sketches/hints towards solutions.

Problem 1. Find the first three nonzero terms in the MacLaurin series of $\arctan(x) \cdot \ln(1+x)$ and $\tan x$.

Solution. $\arctan(x) \cdot \ln(1+x) = x^2 - \frac{x^3}{2} - \frac{x^5}{12} + \dots$ and $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{12} + \dots$. For $\tan x$ either compute derivatives, do long division for $\frac{\sin x}{\cos x}$, or set $\tan x = \sum_{n=0}^{\infty} a_n x^n$, and then solve $\cos(x) \sum_{n=0}^{\infty} a_n x^n = \sin(x)$ by multiplying out the first couple terms on the left and comparing coefficients.

Problem 2. Use Taylor series to compute the limits:

a) $\lim_{x \to 0} \frac{4x^5 + x^6 - 3x^7}{x - \frac{x^3}{3!} - \sin(x)}$,

b) $\lim_{x \to 1} \frac{(x-1)\ln x}{\sin^2(\pi x)}$,

c) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$,

d) $\lim_{x \to 0} \frac{\sqrt{1+x} - \cos(x) - \frac{1}{2}\sin(x)}{\tan^2(x)}$.

Solution.

a) -480.

b) $\frac{1}{\pi^2}$.

c) $\frac{1}{3}$.

d) $\frac{3}{8}$.

Problem 3. Use a second order Taylor polynomial to approximate $\cos(0.1)$ and compute the error bound. What changes if you use T_3 instead?

Solution. Use center $a = 0$. $T_2(x) = 1 - \frac{x^2}{2}$, so the approximation is $T_2(0.1) = 1 - \frac{1}{200} = 0.995$. The error bound for $n = 2$ is $\frac{1}{3!}0.1^3 = \frac{1}{6000}$ (we can choose $M = 1$). If we choose T_3 instead the approximation doesn't change since $T_2 = T_3$ in this case, however we get a better error bound: $\frac{1}{4!}0.1^4 = \frac{1}{24 \cdot 10000}$. So we basically get a better error bound "for free".

Problem 4. Approximate $\sqrt[3]{28}$ with an error of at most 0.001. Is the approximation larger or smaller than $\sqrt[3]{28}$?

Solution. Let $f(x) = \sqrt[3]{x}$. We use Taylor polynomials centered at $a = 27$ to get an approximation. We have

$$f^{(n)}(x) = \frac{1}{3}\left(\frac{1}{3} - 1\right)\cdots\left(\frac{1}{3} - n + 1\right)x^{\frac{1}{3}-n}.$$

So for $n \geq 1$ we can the bound for $x \in [27, 28]$:

$$\begin{aligned} \left|f^{(n)}(x)\right| &= \left|\frac{1}{3}\left(\frac{1}{3} - 1\right)\cdots\left(\frac{1}{3} - n + 1\right)x^{\frac{1}{3}-n}\right| \\ &\leq \left|\frac{1}{3}\left(\frac{1}{3} - 1\right)\cdots\left(\frac{1}{3} - n + 1\right)27^{\frac{1}{3}-n}\right| \\ &= \left|\frac{1}{3}\left(\frac{1}{3} - 1\right)\cdots\left(\frac{1}{3} - n + 1\right)\right| 3 \cdot 27^{-n} =: M_n \end{aligned}$$

This is because for $n \geq 1$ we have $\frac{1}{3} - n < 0$, so the function $x^{\frac{1}{3}-n}$ is decreasing and takes on its maximum at the left endpoint $x = 27$ of the interval. So we want to find n such that

$$\frac{M_{n+1}}{(n+1)!}(28 - 27)^{n+1} \leq 0.001.$$

Trying out $n = 0, 1$ shows that $n = 1$ works, then $\frac{M_{n+1}}{(n+1)!}(28-27)^{n+1} = \frac{1}{3 \cdot 27^2} < \frac{1}{1000}$. So the first order Taylor polynomial is sufficient. We have

$$T_1(x) = f(27) + f'(27)(x - 27) = 3 + \frac{1}{27}(x - 27),$$

so our approximation is $T_1(28) = 3 + \frac{1}{27}$. The true value of $\sqrt[3]{28}$ must be smaller than this since $f(x) = \sqrt[3]{x}$ is concave, hence the tangent line of f at $(27, 3)$ lies above the graph of f, so we have the inequality $T_1(x) \geq f(x)$. Alternatively you can compute $\left(3 + \frac{1}{27}\right)^3 = 3^3 + 3 \cdot 3^2 \cdot \frac{1}{27} + 3 \cdot 3 \cdot \frac{1}{27^2} + \frac{1}{27^3} = 28 + 3 \cdot 3 \cdot \frac{1}{27^2} + \frac{1}{27^3} > 28$.

Note: Using a calculator we find that $\sqrt[3]{28} - \left(3 + \frac{1}{27}\right) \approx -0.00045$.

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